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CONVERGENCE OF FOURIER SERIES

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Convergence of the Fourier series of functions from the class $L\log^+ L\log^+\log^+\log^+ L$ is proved.

1. Introduction

Let $\varphi : [0,+\infty) \to [0,+\infty)$ be a given non-decreasing function, $\varphi(0) = 0$. Set $T := [0, 2\pi)$ and denote by $\varphi(L) = \varphi(L)_T$ the set of measurable 2π-periodic functions f such that

$$\int_T \varphi(|f(t)|)dt < \infty.$$

Let $S_n(f,x)$ denote the nth partial sum of Fourier series of the function f at the point x. Introduce the function

$$Mf(x) := \sup_{n\geq 1} |S_n(f,x)|, \quad x \in T.$$

Carleson [1] proved that the Fourier series of any function $f \in L^2$ converges almost everywhere (a.e.). This fact was extended by Hunt [2]. He proved a.e. convergence of the Fourier series for functions $f \in L^p$, $p > 1$, and for $f \in L(\log^+ L)^2$. The basic result of [2] is the following estimate for the characteristic function χ_F of an arbitrary set $F \subset T$.

$$(1) \qquad m\{x \in T : M\chi_F(x) > y\} \leq (B_p)^p y^{-p} mF, \quad y > 0, \quad 1 < p < \infty,$$

where $B_p \le const.\ p^2/(p-1)$. It follows from (1) (see [3, p. 563]) that

$$(2) \qquad m\{x \in T : M\chi_F(x) > y\} \le C\frac{1}{y}\log(\frac{1}{y})mF \quad \text{for } 0 < y < \frac{1}{2},$$

with some constant C. Using (2) Sjölin [3] proved the almost everywhere convergence of Fourier series for functions f from the class $L(\log^+ L)(\log^+ \log^+ L)$. In this paper we prove the following.

Theorem. *If $f \in L(\log^+ L)(\log^+ \log^+ \log^+ L)$, then the Fourier series of f converges almost everywhere.*

We obtain this result from (2) making use of the method from [3] (see our Lemma 3) and "approximation" of $f(x)$ in the sense of Lemma 2 (point a) below.

2. Notation

For the sake of simplicity, we set $a_k := 2^{2^k}$ ($a_0 = 2$, $a_1 = 4$, $a_2 = 16, \ldots$). Note that $a_k = (a_{k-1})^2$, $k = 1, 2, \ldots$. For any measurable function $f(x) \ge 0$ we define

$$J(f) := \int_T (f(x)\log^+ f(x) \log^+ \log^+ \log^+ f(x))\, dx,$$

where $\log u = \log_2 u$ and $y^+ := \max\{0, y\}$ $(-\infty \le y < +\infty)$. Remark that for $u \ge 16$ the function $\psi(u) := \log u \log\log\log u$ is increasing and $\psi(u^2) \le 4\psi(u)$. For $n = 1, 2, \ldots$, set

$$M_n f(x) := \max_{1 \le m \le n} |S_m(f, x)|, \quad x \in T.$$

In what follows C will denote the constant from estimate (2).

3. Preliminary lemmas

Lemma 1. *Let $\varepsilon > 0$, $a > 0$, $n \in \mathbf{N}$. Assume that $G \subset T$, G is a measurable set and f is a measurable function such that $0 \le f(x) \le a$ for $x \in G$, and $f(x) = 0$ for $x \notin G$. Then there exists a set $F \subset G$ which satisfies the following conditions:*

a) $m \le n$ implies $\|S_m(f, x) - S_m(a\chi_F, x)\|_C < \varepsilon$,

b) $\int_G a\chi_F(x)\,dx = am(F) = \int_G f(x)\,dx$.

Proof. Denote $I = \int\limits_G f(x)\,dx$. If $I = 0$, then the lemma is obviously true. Assume that $I \neq 0$. Let us choose l such that

$$\frac{2\pi}{l} < \frac{\varepsilon}{2I(n+1)^2}. \tag{3}$$

Introduce the notations

$$A_i := 2\pi i/l,\ i = 0, 1, \ldots, l, \quad \Delta_i := [A_{i-1}, A_i),\ i = 1, \ldots, l.$$

Denote

$$G^i := G \cap \Delta_i, \quad I_i := \int_{G^i} f(x)\,dx.$$

It is easy to see that $G^i \cap G^j = \emptyset$, $i \neq j$, $G = \cup_{i=1}^{l} G^i$, $I_i \leq am(G_i)$ and

$$I = \sum_{i=1}^{l} I_i. \tag{4}$$

We can find a set $F^i \subset G^i$ such that $m(F^i) = I_i/a$ and hence,

$$\int\limits_{\Delta_i} a\chi_{F^i}(x)\,dx = \int\limits_{G^i} a\chi_{F^i}(x)\,dx = I_i = \int\limits_{\Delta_i} f(x)\,dx. \tag{5}$$

Set $F := \cup_{i=1}^{l} F^i$. Using (4) and (5), we obtain

$$\int_G a\chi_F(x)\,dx = \sum_{i=1}^{l} \int_{G^i} a\chi_F(x)\,dx = \sum_{i=1}^{l} \int_{G^i} a\chi_{F^i}(x)\,dx = \sum_{i=1}^{l} I_i = I,$$

and thus b) holds. Let us verify a). For $m \leq n$ and $x \in T$, we have

$$|S_m(f,x) - S_m(a\chi_F, x)| = |\int\limits_T D_m(x-t)(f(t) - a\chi_F(t))dt|$$

$$= |\sum_{i=1}^{l} \int\limits_{\Delta_i} D_m(x-t)(f(t) - a\chi_F(t))dt|$$

$$= |\sum_{i=1}^{l} \int\limits_{\Delta_i} (D_m(x-t) - D_m(x-A_i))(f(t) - a\chi_F(t))dt|.$$

Using these equalities, as well as (5), (3) and the classical Bernstein inequality $\|D'_m\|_C \le m\|D_m\|_C$, we obtain

$$|S_m(f,x) - S_m(a\chi_F, x)| \le \sum_{i=1}^{l} 2 \max_{t\in\Delta_i} |D_m(x-t) - D_m(x-A_i)| \int_{G^i} f(t)\,dt$$

$$\le \sum_{i=1}^{l} 2(m+1)^2 \frac{2\pi}{l} I_i \le \sum_{i=1}^{l} 2(m+1)^2 I_i \frac{\varepsilon}{2I(n+1)^2} \le \varepsilon.$$

Lemma 1 is proved.

Lemma 2. *Let $\psi : [0,+\infty) \to [0,+\infty)$ be a non-decreasing function such that $\psi(u^2) \le 4\psi(u)$ for $u \ge 16$. Define $\varphi(u) := u\psi(u)$, $u \ge 0$. Let $\varepsilon > 0$, $n \in \mathbf{N}$, $f(x) \in \varphi(L)$ and for any $x \in T$, $f(x) \ge 16$ or $f(x) = 0$. Then there exists a sequence of sets F_k, $k = 2,3,\ldots$, $F_k \subset T, F_i \cap F_j = \emptyset$, $i \ne j$, such that the function $g(x) := \sum_{k=2}^{\infty} a_k \chi_{F_k}(x)$ satisfies the following conditions:*

a) If $m \le n$, then $\|S_m(f,x) - S_m(g,x)\|_C < \varepsilon$,

b) $\int_T \varphi(g(x))\,dx \le 4 \int_T \varphi(f(x))\,dx$.

Proof. Fix $\varepsilon > 0$, $n \in \mathbf{N}$ and a function $f \in \varphi(L)$ such that $f(x) \ge 16$. Define $G_k := \{x \in T : a_{k-1} < f(x) e a_k\}$, $k = 2,3,\ldots$. Obviously $G_i \cap G_j = \emptyset$ for $i \ne j$. Let

$$f_k(x) := \begin{cases} f(x), & x \in G_k\,, \\ 0, & x \notin G_k. \end{cases}$$

Since $f(x) = \sum_{k=2}^{\infty} f_k(x)$ and $\varphi(0) = 0$, we have

$$\int_T \varphi(f(x))\,dx = \sum_{k=2}^{\infty} \int_{G_k} \varphi(f(x))\,dx = \sum_{k=2}^{\infty} \int_{G_k} \varphi(f_k(x))\,dx. \tag{6}$$

In view of Lemma 1, for every $f_k(x)$, $k = 2,3,\ldots$, there exists$F_k \subset G_k$ such that

$$\|S_m(f_k,x) - S_m(a_k\chi_{F_k},x)\|_C < \frac{\varepsilon}{2^k} \tag{7}$$

and

$$\int_{G_k} f_k(x)\,dx = a_k m(F_k) = \int_{G_k} a_k \chi_{F_k}(x)\,dx. \tag{8}$$

Note that

$$\int_T \varphi(g(x))\,dx = \sum_{k=2}^{\infty} \int_{G_k} \varphi(g(x))\,dx = \sum_{k=2}^{\infty} \int_{G_k} \varphi(a_k\chi_{F_k}(x))\,dx. \tag{9}$$

Using (8) and the assumptions on the function φ, we obtain

$$\int_{G_k} \varphi(a_k\chi_{F_k}(x))\,dx = \int_{G_k} \varphi(a_k)\chi_{F_k}(x)\,dx = \psi(a_k)a_k m(F_k)$$

$$= \int_{G_k} f_k(x)\psi(a_k)\,dx = \int_{G_k} f_k(x)\psi((a_{k-1})^2)\,dx \le \int_{G_k} f_k(x)\psi((f_k(x))^2)\,dx$$

$$\le 4\int_{G_k} f_k(x)\psi(f_k(x))\,dx = 4\int_{G_k} \varphi(f_k(x))\,dx, \quad k = 2,3,\dots . \tag{10}$$

It follows from (9), (10) and (6) that

$$\int_T \varphi(g(x))\,dx \le 4\int_T \varphi(f(x))\,dx.$$

Let $m \le n$. From (7) we obtain

$$\begin{aligned} \|S_m(f,x) - S_m(g,x)\|_C &= \|S_m(\sum_{k=2}^{\infty}(f_k - a_k\chi_{F_k}), x)\|_C \\ &= \|\sum_{k=2}^{\infty} S_m(f_k - a_k\chi_{F_k}, x)\|_C \\ &\le \sum_{k=2}^{\infty} \|S_m(f_k - a_k\chi_{F_k}, x)\|_C \\ &\le \sum_{k=2}^{\infty} \frac{\varepsilon}{2^k} < \varepsilon. \end{aligned}$$

Lemma 2 is proved.

Lemma 3. *Assume that $g(x) = \sum_{n=2}^{\infty} a_n\chi_{F_n}(x), F_i \cap F_j = \emptyset$, $i \notin j$, $J(g) < \frac{1}{C}$. Then there exists a set B with $mB \le 2$ such that $Mg(x) \le 13$ for $x \notin B$.*

Proof. Denote

$$E_n := \{x \in T : M\chi_{F_n}(x) > a_n^{-1}\}, \quad E = \cup_{n=2}^{\infty} E_n,$$

$$K_n := \{x \in T : a_n^{-1}n^{-2} < M\chi_{F_n}(x) \le a_n^{-1}\},$$

$$L_n := \{x \in T : M\chi_{F_n}(x) \le a_n^{-1}n^{-2}\}. \tag{11}$$

It follows from (2) and (1) that $mE_n \le Ca_n \log a_n mF_n$, and hence

$$mE \le C\sum_{n=2}^{\infty} a_n \log a_n mF_n$$

$$= C\sum_{n=2}^{\infty}\int_T a_n \log a_n \chi_{F_n}(x)\,dx$$
$$\leq C\sum_{n=2}^{\infty}\int_T a_n \log a_n \log\log\log a_n \chi_{F_n}(x)\,dx.$$

According to the definition of $J(g)$ and the assumptions of the lemma,

$$mE \leq CJ(g) < 1. \tag{12}$$

Consider the function $Mg(x) = M(\sum_{n=2}^{\infty} a_n\chi_{F_n}(x))$ outside E. We have

$$\int_{T\setminus E} M(\sum_{n=2}^{\infty} a_n\chi_{F_n}(x))\,dx \leq \sum_{n=2}^{\infty}\int_{T\setminus E_n} a_n M(\chi_{F_n}(x))\,dx$$

$$\leq \sum_{n=2}^{\infty} a_n \int_{K_n} M(\chi_{F_n}(x))\,dx + \sum_{n=2}^{\infty} a_n \int_{L_n} M(\chi_{F_n}(x))\,dx. \tag{13}$$

It follows from the definition of L_n that

$$\sum_{n=2}^{\infty} a_n \int_{L_n} M(\chi_{F_n}(x))\,x \leq \sum_{n=2}^{\infty}\frac{2\pi}{n^2} < 5. \tag{14}$$

We shall now prove the estimate

$$a_n\int_{K_n} M(\chi_{F_n}(x))\,dx \leq 8C\int_{F_n} g(x)\log g(x)\log\log\log g(x)\,dx. \tag{15}$$

In order to do this, let $\mu_n(\lambda) = m\{x \in T : M(\chi_{F_n}(x)) > \lambda\}$. Then

$$a_n\int_{K_n} M(\chi_{F_n}(x))\,dx = -a_n\int_{a_n^{-1}n^{-2}}^{a_n^{-1}} \lambda d\mu_n(\lambda)$$
$$= -a_n\lambda\mu_n(\lambda)\Big|_{a_n^{-1}n^{-2}}^{a_n^{-1}} + a_n\int_{a_n^{-1}n^{-2}}^{a_n^{-1}} \mu_n(\lambda)d\lambda.$$

Using (2), we obtain

$$a_n\int_{K_n} M(\chi_{F_n}(x))\,dx \leq C\log(a_n n^2)a_n mF_n + Ca_n\int_{a_n^{-1}n^{-2}}^{a_n^{-1}}\frac{\log(1/\lambda)}{\lambda}d\lambda\, mF_n$$

$$\begin{aligned}
&\le\ Ca_n\log(a_n n^2)mF_n+\frac{C}{2}a_n[(\log(a_n n^2))^2-(\log a_n)^2]mF_n\\
&\le\ Ca_n\log(a_n n^2)mF_n+Ca_n[\log(a_n n^2)-\log a_n]\log(a_n n^2)mF_n\\
&\le\ 8Ca_n\log a_n\log n\,mF_n\\
&=\ 8Ca_n\log a_n\log\log\log a_n\,mF_n
\end{aligned}$$

and (15) follows. From (13), (14) and (15), we have

$$\int\limits_{T\setminus E} M(g(x))\,dx \le 8C\,J(g)+5\le 13.$$

Let $H=\{x\in T\setminus E : Mg(x)>13\}$. It is easy to see that

$$mH<1. \tag{16}$$

Now we can define the set B. Let $B:=E\cup H$. According to (12) and (16), $mB<2$, and $Mg(x)\le 13$ for $x\notin B$. The lemma is proved.

4. A property of sequences of operators

Let $\varphi:[0,+\infty)\to[0,+\infty)$ be a function such that: (i) $\varphi(0)=0$, $\varphi(u)$ is convex and increasing on$[0,\infty)$, (ii) $\varphi(u^{\frac{1}{2}})$ is a concave function of u, $0\le u<\infty$. We need a certain proposition which follows from a theorem of Stein [4]. Before formulating the result, let us first introduce some definitions. We say that an operator V is of type (φ,φ) if there exists a constant $A>0$ so that

$$\int\limits_T \varphi(|V(f,x)|)\,dx\le\int\limits_T\varphi(A|f(x)|)\,dx.$$

The operator V *commutes with translations* if

$$V(f(\cdot+s),x)=V(f(\cdot),x+s).$$

Theorem A. [4, p. 154, Theorem 3]. *Let V_m be a sequence of linear operators, each of type (φ,φ), and which commute with translations. Let φ satisfy the conditions (i) and (ii). Suppose that for every $f\in\varphi(L)$ we have $\limsup_{m\to\epsilon ty}|V_m(f,x)|<\infty$, for x in some set of positive measure. Let $V^*(f,x)=\sup_{m>1}|V_m(f,x)|$. Then there exists a constant A such that*

$$m\{x:V^*(f,x)>\alpha\}\le\int\limits_T\varphi(\frac{A}{\alpha}|f(x)|)\,dx,\quad \alpha>0.$$

Lemma 4. *Let φ satisfy (i) and (ii). For $n \in \mathbf{N}$, $a \in \mathbf{R}$, $A_n \in L^\infty(T)$, assume that*

$$U_n(f,x) := \int_T A_n(x-t)f(t)dt + af(x)$$

is a sequence of operator such that: 1) There exists a set P of functions which is dense in $\varphi(L)$ and such that, for any $p \in P$,

$$\limsup_{n\to\infty} |U_n(p,x)| = 0 \quad a.e.; \tag{17}$$

2) *For every $f \in \varphi(L)$, the estimate*

$$\sup_{n\geq 1} |U_n(f,x)| < \infty$$

holds for x in some set of positive measure. Then for every $f \in \varphi(L)$,

$$\limsup_{n\to\infty} |U_n(f,x)| = 0$$

for almost every x.

Proof. Denote

$$U^*(f,x) := \sup_{n\geq 1} |U_n(f,x)|,$$

$$U(f,x) := \limsup_{n\to\infty} |U_n(f,x)|.$$

Conditions (i), (ii) imply: $\alpha^2\varphi(u) \leq \varphi(\alpha u) \leq \alpha\varphi(u)$ $(0 \leq \alpha \leq 1)$. Applying this and Jensen's inequality, it is easy to verify that the sequence U_n satisfies the conditions of Theorem A. On the basis of this theorem we conclude that there exists $A > 0$ so that

$$m\{x \in T : U(f,x) > \alpha\} \leq m\{x \in T : U^*(f,x) > \alpha\}$$

$$\leq \int_T \varphi(\frac{A|f(x)|}{\alpha})\,dx \leq \frac{A}{\alpha}\int_T \varphi(|f(x)|)\,dx,$$

for $\alpha > A$ and $f \in \varphi(L)$. Hence there exists $K > 0$ such that for all $f \in \varphi(L)$,

$$\int_T (U(f,x))^{\frac{1}{2}}\,dx \leq K\int_T \varphi(|f(x)|)\,dx + K. \tag{18}$$

The proof of (18) is not difficult due to the representation of the left-hand part of (18) in terms of$\mu(\lambda) = m\{x \in T : U(f,x) > \lambda\}$. Let $\theta > 0$ and

$f \in \varphi(L)$. According to the density of the set P, there exists $p_0(x) \in P$ such that

$$\int_T \varphi(|\theta^2 f(x) - p_0(x)|)\, dx < 1. \tag{19}$$

Using consecutively (17), (18) and (19), we get

$$\begin{aligned} \theta \int_T (U(f,x))^{\frac{1}{2}}\, dx &= \int_T |U(\theta^2 f, x) - U(p_0, x)|^{\frac{1}{2}}\, dx \\ &\leq \int_T (U(\theta^2 f - p_0, x))^{\frac{1}{2}}\, dx \\ &\leq K \int_T \varphi(|\theta^2 f(x) - p_0(x)|)\, dx + K \leq 2K, \end{aligned}$$

and thus

$$\int_T (U(f,x))^{\frac{1}{2}}\, dx \leq \frac{2K}{\theta} \quad \text{for any } \theta > 0.$$

Let $\theta \to \infty$. We get $\int_T (U(f,x))^{\frac{1}{2}}\, dx = 0$. Therefore $U(f,x) = 0$ a.e. in T. Lemma 4 is proved.

Corollary. *Assume that for every $f \in L \log^+ L \log^+ \log^+ \log^+ L$ we have $\sup_{n\geq 1} |S_n(f,x)| < \infty$ for x in some set of positive measure. Then the Fourier series of any $f \in L \log L\ og^+ \log^+ \log^+ L$ converges a.e. .*

To prove this corollary we just apply Lemma 4 for $U_n(f,x) := S_n(f,x) - f(x)$ with P being the set of all trigonometric polynomials. Let us note also that there exists a function φ, satisfying conditions (i) and (ii), such that $\varphi(L) = L \log^+ L \log^+ \log^+ \log^+ L$.

5. Proof of the theorem

First we assume that $f : T \to \{0\} \cup [16, +\infty)$ and $J(f) < 1/(4C)$. Let $n \in \mathbf{N}$. Then, according to Lemma 2, there exists a function $g(x) = g_n(x) = \sum_{k=2}^{\infty} a_k \chi_{F_k}(x)$ such that

$$J(g) \leq 4J(f) \leq \frac{1}{C}$$

and

$$\|M_n f - M_n g_n\|_C \leq \|M_n(f - g_n)\|_C = \max_{1\leq m\leq n} \|S_m(f - g_n, \cdot)\|_C < 1. \tag{20}$$

It follows from (20) that for all $x \in T$, we have $M_n f(x) \leq M_n g_n(x) + 1$. Therefore

$$\{x \in T : M_n f(x) > 14\} \subset \{x \in T : M_n g_n(x) > 13\}. \tag{21}$$

According to (21) and Lemma 3,

$$m\{x \in T : M_n f(x) > 14\} \leq m\ x \in T : M_n g_n(x) > 13\} \leq 2.$$

Hence

$$m\{x \in T : Mf(x) > 14\} = \lim_{n\to\infty} m\{x \in T : M_n f(x) > 14\} \leq 2. \tag{22}$$

Using (22), the convergence a.e. of Fourier series of bounded functions and linearity of the set $\varphi(L) = L\log^+ L \log^+ \log^+ \log^+ L$, we can prove that for every $f \in \varphi(L)$

$$\limsup_{n\to\infty} |S_n(f,x)| < \infty$$

on a set of positive measure. According to the corollary of Lemma 4, this fact implies almost everywhere convergence of the Fourier series of functions from $L\log^+ L \log^+ \log^+ \log^+ L$. The theorem is proved.

Remark. Clearly, the construction proposed here can be applied to improve the well-known Sjölin result on a.e. convergence of multiple Fourier series, namely, to prove that the condition $f \in L(\log^+ L)^d \log^+ \log^+ \log^+ L(T^d)$ is sufficient for such a convergence. We shall publish this result soon in another paper.

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